On Bethe vectors for the XXZ model at roots of unity

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Recently in a series of papers [DFM], [FM1], [FM2] the Bethe ansatz equations and Bethe vectors for the six-vertex model with the anisotropy commensurable with π or, as usually said, at roots of unity, were studied. In that case the spectrum of the transfer-matrix, which is a generating function of the standard commuting conservation laws in the model, becomes highly degenerate. In [FM2] a construction of creation operators, responsible for appearance of the Bethe vectors with the same eigenvalues of the transfer-matrix, was suggested in the framework of the algebraic Bethe ansatz. In the note we extend that construction to the case of the inhomogeneous arbitrary spin XXZ model. Even for the case of six-vertex model the proof of the main formulae given in the note is simpler than the original proof in [FM2].

The detailed exposition of the algebraic Bethe ansatz method can be found in [KBI].

The notation used in the note does not coincide with those of [FM2] and [KBI], however a reader can easily establish the correspondence.

1. Consider the inhomogeneous XXZ model on the N-vertex lattice with the anisotropy γ and the quasiperiodic boundary conditions. Let $q = e^{i\gamma}$. Let $\ell_1, \dots, \ell_N$ be the spins of representations at vertices, $z_1, \dots, z_N$ — the inhomogeneity parameters, and κ — the quasiperiodicity parameter, the periodic boundary conditions corresponding to $\kappa = 1$. We assume that $q^2 \neq 1$ and $z_i \neq 0$, $2\ell_i \in \mathbb{Z}_{>0}$ for all $i = 1, \dots, N$.

For any x set $q^x = e^{i\gamma x}$ and $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}} = \dfrac{\sin(\gamma x)}{\sin\gamma}$.

The XXZ model is described by the L-operator

$$L(u) = \begin{pmatrix} uq^H - q^{-H} & u(q-q^{-1})F \\ (q-q^{-1})E & uq^{-H} - q^H \end{pmatrix}$$

where the elements E, F, H are generators of $U_q(\mathfrak{sl}_2)$:

$$[H, E] = E, \qquad [H, F] = -F, \qquad [E, F] = [2H]_q.$$

Define a representation of $U_q(\mathfrak{sl}_2)$ of nonnegative integral or semiintegral spin ℓ in the space $V_\ell = \bigoplus\limits_{r=0}^{2\ell} \mathbb{C} v_r$ as follows:

$$(1.1)\qquad E v_r = [r]_q v_{r-1}, \qquad F v_r = [2\ell - r]_q v_{r+1}, \qquad H v_r = (\ell - r) v_r.$$

It is irreducible, if $q^{2k} \neq 1$ for any $k = 1, \dots, 2\ell$. Recall that for such q the algebra

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$U_q(\mathfrak{sl}_2)$ has a unique up to equivalence irreducible representation of dimension $2\ell+1$. In what follows it is important for us that formulae (1.1) are analytic in q, though their explicit form is not quite essential.

Entries of the mondromy matrix

$$z_1 \dots z_N \, L_N(u/z_N) \dots L_1(u/z_1) \;=\; \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

are polynomials in u and Laurent polynomials in $q^{1/2}$ with values in $\mathrm{End}\,(V_{\ell_1} \otimes \dots \otimes V_{\ell_N})$. If necessary, their dependence on q will be shown explicitly, for instance, $A(u;q)$.

The transfer-matrix $T(u) = A(u)+\kappa D(u)$ commutes with the operator $H_\Sigma = H_1 + \dots + H_N$ and is a generating function of the standard commuting conservation laws. The algebraic Bethe ansatz gives a way to find eigenvectors and eigenvalues of the transfer-matrix, cf. Proposition 1.1. Recall the main points of the technique.

The algebraic Bethe ansatz is based on the following commutation relations for entries of the monodromy matrix:

$$\big[B(u), B(v)\big] \;=\; 0\,, \tag{1.2}$$

$$(u-v)\,A(u)B(v) \;=\; (uq^{-1}-vq)\,B(v)A(u) \,+\, v\,(q-q^{-1})\,B(u)A(v)\,, \tag{1.3}$$

$$(u-v)\,D(u)B(v) \;=\; (uq-vq^{-1})\,B(v)D(u) \,-\, v\,(q-q^{-1})\,B(u)D(v)\,, \tag{1.4}$$

and formulae for the action of $A(u)$ and $D(u)$ on the vector $v_0 \otimes \dots \otimes v_0$:

$$A(u)\,v_0 \otimes \dots \otimes v_0 \;=\; \mathcal{A}(u)\,v_0 \otimes \dots \otimes v_0\,, \tag{1.5}$$

$$\mathcal{A}(u) \;=\; \prod_{i=1}^{N} (uq^{\ell_i} - z_i q^{-\ell_i})\,,$$

$$D(u)\,v_0 \otimes \dots \otimes v_0 \;=\; \mathcal{D}(u)\,v_0 \otimes \dots \otimes v_0\,, \tag{1.6}$$

$$\mathcal{D}(u) \;=\; \prod_{i=1}^{N} (uq^{-\ell_i} - z_i q^{\ell_i})\,.$$

Let $|t_1,\dots,t_k\rangle = B(t_1)\dots B(t_k)\,v_0 \otimes \dots \otimes v_0$. Formulae (1.2)–(1.6) imply that the vector $|t_1,\dots,t_k\rangle$ is a symmetric function of $t_1,\dots,t_k$, while the operators H_Σ, $A(u)$ and $D(u)$ act on it as follows:

$$H_\Sigma\,|t_1,\dots,t_k\rangle \;=\; (\ell_1+\dots+\ell_N-k)\,|t_1,\dots,t_k\rangle\,,$$

$$\begin{aligned} A(u)\,|t_1,\dots,t_k\rangle \;=\; & \mathcal{A}(u) \prod_{a=1}^{k} \frac{uq^{-1}-t_a q}{u-t_a}\;|t_1,\dots,t_k\rangle \,+ \\ & +\,(q-q^{-1}) \sum_{a=1}^{k} \frac{t_a}{u-t_a}\,\mathcal{A}(t_a) \prod_{\substack{b=1\\ b\neq a}}^{k} \frac{t_a q^{-1}-t_b q}{t_a-t_b}\;|u,t_1,\dots,\widehat{t_a},\dots,t_k\rangle\,, \end{aligned} \tag{1.7}$$

$$
\begin{aligned}
(1.8)\qquad D(u)\,|t_1,\dots,t_k\rangle \;&=\; \mathcal{D}(u)\prod_{a=1}^{k}\frac{uq-t_aq^{-1}}{u-t_a}\;|t_1,\dots,t_k\rangle\;-\\
&-\;(q-q^{-1})\sum_{a=1}^{k}\frac{t_a}{u-t_a}\,\mathcal{D}(t_a)\prod_{\substack{b=1\\ b\neq a}}^{k}\frac{t_aq-t_bq^{-1}}{t_a-t_b}\;|u,t_1,\dots,\widehat{t_a},\dots,t_k\rangle\,.
\end{aligned}
$$

The system of Bethe ansatz equations for rapidities $t_1,\dots,t_k$ has the form:

$$
(1.9)\qquad \mathcal{A}(t_a)\prod_{\substack{b=1\\ b\neq a}}^{k}(t_a-t_bq^2)\;=\;\kappa\,\mathcal{D}(t_a)\prod_{\substack{b=1\\ b\neq a}}^{k}(t_aq^2-t_b)\,,\qquad\qquad a=1,\dots,k\,.
$$

We do not distinguish solutions of this system which are obtained from each other by permutations of the variables $t_1,\dots,t_k$. We say that a solution $t_1,\dots,t_k$ contains a point u if $u\in\{t_1,\dots,t_k\}$. A solution $t_1,\dots,t_k$ of system (1.9) is called *offdiagonal* if $t_a\neq t_b$ for all $a,b=1,\dots,k$. The vector $|t_1,\dots,t_k\rangle$ is called the Bethe vector if $t_1,\dots,t_k$ is an offdiagonal solution of the Bethe ansatz equations. The vector $v_0\otimes\dots\otimes v_0$ is the Bethe vector corresponding to the empty set of rapidities by convention.

Proposition 1.1. *Let $t_1,\dots,t_k$ be an offdiagonal solution of system* (1.9)*. Then*

$$
\begin{aligned}
(1.10)\qquad T(u)\,|t_1,\dots,t_k\rangle \;&=\\
&=\;\Bigl(\mathcal{A}(u)\prod_{a=1}^{k}\frac{uq^{-1}-t_aq}{u-t_a}\;+\;\kappa\mathcal{D}(u)\prod_{a=1}^{k}\frac{uq-t_aq^{-1}}{u-t_a}\Bigr)\,|t_1,\dots,t_k\rangle\,.
\end{aligned}
$$

The statement follows from formulae (1.7) and (1.8).

Henceforth we assume that $q^{2\ell_i}z_i\neq q^{-2\ell_j}z_j$ for any $i,j=1,\dots,N$, that is, the polynomials $\mathcal{A}(u)$ and $\mathcal{D}(u)$ are coprime. Furthermore, we assume that $q^{2r}\neq 1$ for all $r=1,\dots,\max(2\ell_1,\dots,2\ell_N)$, in other words, that all the representations $V_{\ell_1},\dots,V_{\ell_N}$ are irreducible.

A solution $t_1,\dots,t_k$ of system (1.9) is called *admissible* if $t_a\neq 0$ and $t_a\neq q^2t_b$ for all $a,b=1,\dots,k$. Simple analysis of system (1.9) yields the following properties of admissible solutions

Lemma 1.2. *Any admissible solution of system* (1.9) *does not contain the points $q^{\pm 2\ell_1}z_1,\dots,q^{\pm 2\ell_N}z_N$.*

Lemma 1.3. *Let $q^{2m}\neq 1$ for all $m=1,\dots,k$.*

a) *If a solution of system* (1.9) *does not contain any of the points $q^{2\ell_1}z_1,\dots,q^{2\ell_N}z_N$, then it is admissible.*

b) *If a solution of system* (1.9) *does not contain any of the points $q^{-2\ell_1}z_1,\dots,q^{-2\ell_N}z_N$, then it is admissible.*

We say that the points $z_1,\dots,z_N$ are *well separated*, if $q^{2(r-\ell_i)}z_i\neq q^{2(s-l_j)}z_j$ for all $r=0,\dots,2\ell_i$, $s=0,\dots,2\ell_j$, $i,j=1,\dots,N$.

Theorem 1.4. [TV, Theorems 4.2 and 5.1] *Let the points $z_1,\dots,z_N$ be well separated. Then for generic κ all admissible offdiagonal solutions of system* (1.9) *are nondegenerate, the number of distinct admissible offdiagonal solutions of system* (1.9) *equals*

the dimension of the subspace $\{v \in V_{\ell_1} \otimes \ldots \otimes V_{\ell_N} \mid H_\Sigma v = (\ell_1 + \ldots + \ell_N - k)v\}$, *and the corresponding Bethe vectors form a basis of this subspace. If* q *is not a root of unity, then the Bethe vectors corresponding to inadmissible offdiagonal solutions of system* (1.9) *equal zero.*

It is plausible that the assumption of Theorem 1.4 about the points $z_1, \ldots, z_N$ being well separated can be weakened. For example, the statement of the theorem conjecturally remains true for $z_1 = \ldots = z_N$.

2. Assume that q^2 is an M-th root of unity. In principle, for $k \geqslant M$ system (1.9) can have inadmissible solutions of the form $u, uq^2, \ldots, uq^{2M-2}, t_{M+1}, \ldots, t_k$ with arbitrary u; in particular, such solutions are not isolated. The Bethe vector corresponding to such a solution equals zero, because

$$B(u)\,B(uq^2)\ldots B(uq^{2M-2}) \,=\, 0$$

for any u, see [T]. Besides, the sequence $t_{M+1}, \ldots, t_k$ is a solution of system (1.9) for $k - M$ variables.

Theorem 1.4 indicates that to find the spectrum of the transfer-matrix $T(u)$ for generic κ it suffices in general to consider only Bethe vectors corresponding to admissible solutions of system (1.9). At the same time, one can see from the results of [FM2] that for special values of κ inadmissible solutions of system (1.9) mentioned above and corresponding them analogues of Bethe vectors can play essential role in constructing eigenvectors of the transfer-matrix.

Assume that $\kappa = q^{2(p-\ell_1+\ldots+\ell_N)}$ for certain integer p. Let $t_1, \ldots, t_k$ be an offdiagonal solution of system (1.9). The main aim of this note is to construct vectors $\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p$, depending on the parameters $u_1, \ldots, u_m$ and such that

$$H_\Sigma \|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p \,=\, (\ell_1 + \ldots + \ell_N - k - mM)\,\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p\,,$$

$$(2.1) \qquad T(u)\,\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p \,=$$

$$= q^{mM}\Bigl(\mathcal{A}(u)\prod_{a=1}^{k}\frac{uq^{-1} - t_a q}{u - t_a} \,+\, \kappa\mathcal{D}(u)\prod_{a=1}^{k}\frac{uq - t_a q^{-1}}{u - t_a}\Bigr)\,\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p\,,$$

cf. Proposition 3.1. Though the parameters $u_1, \ldots, u_m$ are arbitrary, this does not contradict to the finite-dimensionality of the space of states $V_{\ell_1} \otimes \ldots \otimes V_{\ell_N}$, since the eigenvalue of $T(u)$ does not depend on $u_1, \ldots, u_m$. The construction being suggested generalizes that of [FM2].

The vectors $\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p$ with different m correspond to the same eigenvalue of the operators $q^{\pm H_\Sigma}T(u)$. The respective eigenvalues of the transfer-matrix $T(u)$ coincide, if $q^{mM} = 1$, and can differ by the sign, if $q^{mM} = -1$. Comparison formulae (1.10) and (2.1) shows that the vector $\|t_1, \ldots, t_k; u_1, \ldots, u_m\rangle\!\rangle_p$ can be viewed as an analogue of the Bethe vector corresponding to the inadmissible solution $t_1, \ldots, t_k, u_1, \ldots, u_1 q^{2M-2}, \ldots, u_m, \ldots, u_m q^{2M-2}$ of system (1.9) for $k + mM$ variables.

3. Fix an integer $M > 1$, and let $\gamma_0 = \pi K/M$ for certain K coprime with M. Set $q_0 = e^{i\gamma_0}$ and $\eta = q/q_0$. For any object depending on q we assume that $q = q_0$, unless the dependence is shown explicitly.

Taking into account that $B(u)\,B(uq_0^2)\dots B(uq_0^{2M-2}) = 0$, introduce an operator $\mathbb{B}(u;X)$ depending on a vector $X=(x_1,\dots,x_M)\in\mathbb{C}^M$:

$$\mathbb{B}(u;X) = \lim_{q\to q_0}\bigl((q-q_0)^{-1}B(u\eta^{x_1};q)\,B(uq^2\eta^{x_2};q)\dots B(uq^{2M-2}\eta^{x_M};q)\bigr)\,, \tag{3.1}$$

assuming that $\eta^x\to 1$ as $\eta\to 1$. The operator $\mathbb{B}(u;X)$ does not change under the simultaneous shift of all the parameters $x_1,\dots,x_M$ by the same number. Explicit calculation of the limit in formula (3.1) yields

$$\mathbb{B}(u;X) = \sum_{r=0}^{M-1} B(u)\dots B(uq_0^{2r-2})\bigl(\partial_q B(uq_0^{2r}) + uq_0^{2r-1}(x_{r+1}+2r)\,\partial_u B(uq_0^{2r})\bigr)\times$$
$$\times\,B(uq_0^{2r+2})\dots B(uq_0^{2M-2})\,,$$

which is similar to formula (1.38) in [FM2]. However, it is much more convenient to use formula (3.1) itself, taking the limit $q\to q_0$ only at the end of computation. In particular, relation (1.2) immediately implies that

$$\bigl[\mathbb{B}(u;X),B(v)\bigr] = \bigl[\mathbb{B}(u;X),\mathbb{B}(v;Y)\bigr] = 0$$

for any u,v,X,Y and, therefore, the vector

$$|\!|\!|\,t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle = \mathbb{B}(u_1;X_1)\dots\mathbb{B}(u_m;X_m)\,|t_1,\dots,t_k\rangle$$

is a symmetric function of $t_1,\dots,t_k$, as well as the pairs $(u_1,X_1),\dots,(u_m,X_m)$. It is clear that

$$H_\Sigma\,|\!|\!|\,t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle =$$
$$= (\ell_1+\ldots+\ell_N-k-mM)\,|\!|\!|\,t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle\,.$$

Consider the polynomial $P(u) = \prod_{i=1}^{N}\prod_{r=0}^{2\ell_i-1}(u-z_i\,q_0^{2(\ell_i-r)})$ and the functions

$$Q_n(u;t_1,\dots,t_k) = \frac{u^{k-n}P(u)}{\prod_{a=1}^{k}(u-t_a)(u-t_aq_0^2)}\,,$$

$$F_n(u;t_1,\dots,t_k) = \frac{1}{M}\sum_{r=0}^{M-1}Q_n(uq_0^{2r};t_1,\dots,t_k)\,,$$

$$G_n(u;t_1,\dots,t_k) = \frac{1}{M}\sum_{r=1}^{M-1}r\,Q_n(uq_0^{2r};t_1,\dots,t_k)\,,$$

which satisfy the relations

$$\frac{Q_n(uq_0^2)}{Q_n(u)} = q_0^{2(\ell_1+\ldots+\ell_N-n)} \frac{\mathcal{A}(u)}{\mathcal{D}(u)} \prod_{a=1}^{k} \frac{u-t_a q_0^2}{uq_0^2-t_a},$$

$$F_n(uq_0^2) = F_n(u), \qquad G_n(uq_0^2)-G_n(u) = Q_n(u)-F_n(u).$$

If $t_1,\ldots,t_k$ is an admissible offdiagonal solution of system (1.9), then it is easy to see that the function $F_n(u;t_1,\ldots,t_k)$ is a polynomial in u.

Set $\|t_1,\ldots,t_k;u_1,\ldots,u_m\rangle\!\rangle_n = |\!|\!|t_1,\ldots,t_k;u_1,\ldots,u_m;X_1^{(n)},\ldots,X_m^{(n)}\rangle\!\rangle\!\rangle$ where the vectors $X_i^{(n)} = X^{(n)}(u_i;t_1,\ldots,t_k) = (x_1^{(n)},\ldots,x_M^{(n)})(u_i;t_1,\ldots,t_k)$, $i=1,\ldots,m$, are defined as follows:

$$x_r^{(n)}(u;t_1,\ldots,t_k) = 2\bigl(1-r-G_n(uq^{2r-2};t_1,\ldots,t_k)/F_n(u;t_1,\ldots,t_k)\bigr). \tag{3.2}$$

Proposition 3.1. *Let $\kappa=q_0^{2(p-\ell_1+\ldots+\ell_N)}$ for a certain integer p, and let $t_1,\ldots,t_k$ be an offdiagonal solution of system (1.9) at $q=q_0$. Assume that $F_p(u_i;t_1,\ldots,t_k)\neq 0$ and $u_i^M\neq t_a^M$ for all $a=1,\ldots,k$, $i=1,\ldots,m$. Then*

$$T(u)\,\|t_1,\ldots,t_k;u_1,\ldots,u_m\rangle\!\rangle_p =$$

$$= q_0^{mM}\Bigl(\mathcal{A}(u)\prod_{a=1}^{k}\frac{uq_0^{-1}-t_aq_0}{u-t_a} + \kappa\mathcal{D}(u)\prod_{a=1}^{k}\frac{uq_0-t_aq_0^{-1}}{u-t_a}\Bigr)\|t_1,\ldots,t_k;u_1,\ldots,u_m\rangle\!\rangle_p.$$

Proof. Using formulae (1.7) and (1.8) for generic q and taking the limit $q\to q_0$ we obtain the following formulae for the action of the operators $A(u)$ and $D(u)$ on the vector $|\!|\!|t_1,\ldots,t_k;u_1,\ldots,u_m;X_1,\ldots,X_m\rangle\!\rangle\!\rangle$:

$$\begin{aligned}
A(u)\,|\!|\!|t_1,\ldots,t_k;u_1,\ldots,u_m;X_1,\ldots,X_m\rangle\!\rangle\!\rangle &= \\
= q_0^{mM}\Biggl[\mathcal{A}(u)\prod_{a=1}^{k}\frac{uq_0^{-1}-t_aq_0}{u-t_a}\,|\!|\!|t_1,\ldots,t_k;u_1,\ldots,u_m;X_1,\ldots,X_m\rangle\!\rangle\!\rangle &+ \\
+ (q_0-q_0^{-1})\sum_{a=1}^{k}\frac{t_a}{u-t_a}\,\mathcal{A}(t_a)\prod_{\substack{b=1\\ b\neq a}}^{k}\frac{t_aq_0^{-1}-t_bq_0}{t_a-t_b}\ \times& \\
\times\,|\!|\!|u,t_1,\ldots,\widehat{t_a},\ldots,t_k;u_1,\ldots,u_m;X_1,\ldots,X_m\rangle\!\rangle\!\rangle &- \\
-\sum_{i=1}^{m}\sum_{r=0}^{M-1}\frac{u_iq_0^{2r}}{u-u_iq_0^{2r}}\,(x_{i,r+1}-x_{ir})\,\mathcal{A}(u_iq_0^{2r})\prod_{a=1}^{k}\frac{u_iq_0^{2r-1}-t_aq_0}{u_iq_0^{2r}-t_a}\ \times& \\
\times\prod_{\substack{s=0\\ s\neq r}}^{M-1}B(u_iq_0^{2s})\,|\!|\!|u,t_1,\ldots,t_k;u_1,\ldots,\widehat{u_i},\ldots,u_m;X_1,\ldots,\widehat{X_i},\ldots,X_m\rangle\!\rangle\!\rangle\Biggr]&,
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
(3.4)\quad & D(u)\,\lVert t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle = \\
&= q_0^{mM}\Biggl[\mathcal{D}(u)\prod_{a=1}^{k}\frac{u\,q_0-t_a\,q_0^{-1}}{u-t_a}\,\lVert t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle - \\
&- (q_0-q_0^{-1})\sum_{a=1}^{k}\frac{t_a}{u-t_a}\,\mathcal{D}(t_a)\prod_{\substack{b=1\\ b\neq a}}^{k}\frac{t_a\,q_0-t_b\,q_0^{-1}}{t_a-t_b}\;\times \\
&\qquad\qquad \times\,\lVert u,t_1,\dots,\widehat{t_a},\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle + \\
&+\sum_{i=1}^{m}\sum_{r=0}^{M-1}\frac{u_i\,q_0^{2r}}{u-u_i\,q_0^{2r}}\,(x_{i,r+2}-x_{i,r+1})\,\mathcal{D}(u_i\,q_0^{2r})\prod_{a=1}^{k}\frac{u_i\,q_0^{2r+1}-t_a\,q_0^{-1}}{u_i\,q_0^{2r}-t_a}\;\times \\
&\quad\times\prod_{\substack{s=0\\ s\neq r}}^{M-1}B(u_i\,q_0^{2s})\,\lVert u,t_1,\dots,t_k;u_1,\dots,\widehat{u_i},\dots,u_m;X_1,\dots,\widehat{X_i},\dots,X_m\rangle\!\rangle\!\rangle\Biggr].
\end{aligned}
$$

Here $X_i=(x_{i1},\dots,x_{iM})\,,\ x_{i0}=x_{iM}+2M\,,\ x_{i,M+1}=x_{i1}-2M\,.$

The action of the transfer-matrix $T(u)$ on the vector $\lVert t_1,\dots,t_k;u_1,\dots,u_m;X_1,\dots,X_m\rangle\!\rangle\!\rangle$ produces "unwanted terms" of two kinds, which are respectively generated by the second (ordinary sums) and third (double sums) terms in the right hand sides of formulae (3.3) and (3.4). The unwanted terms of the first kind cancel each other if $t_1,\dots,t_k$ is an offdiagonal solution of system (1.9) at $q=q_0$. Regardless of the cancelation of the unwanted terms of the first kind, the unwanted terms of the second kind cancel each other if

$$
\begin{aligned}
(3.5)\qquad & (x_{i,r+1}-x_{ir})\,\mathcal{A}(u_i\,q_0^{2r})\prod_{a=1}^{k}\frac{u_i\,q_0^{2r}-t_a\,q_0^{2}}{u_i\,q_0^{2r}-t_a} = \\
&= \kappa\,(x_{i,r+2}-x_{i,r+1})\,\mathcal{D}(u_i\,q_0^{2r})\prod_{a=1}^{k}\frac{u_i\,q_0^{2r+2}-t_a}{u_i\,q_0^{2r}-t_a}
\end{aligned}
$$

for all $r=0,\dots,M-1\,,\ i=1,\dots,m\,.$ For $\kappa=q^{2(p-\ell_1+\ldots+\ell_N)}$ these equations hold if $x_{ir}=x_r^{(p)}(u_i)$ for all $r=1,\dots,M\,,\ i=1,\dots,m\,.$ □

For the homogenuous spin-$\frac{1}{2}$ XXZ model (six-vertex model) with the even number of lattice vertices and the periodic boundary conditions: $\ell_1=\ldots=\ell_N=1/2\,,\ z_1=\ldots=z_N=1\,,\ \kappa=1\,,$ Proposition 3.1 was proved in [FM2].

Remark. Consider relations (3.5) for a fixed i as equations for $x_{i1},\dots,x_{iM}$ with the boundary conditions $x_{i0}=x_{iM}+2M\,,\ x_{i,M+1}=x_{i1}-2M\,,$ the variables $u_i,\ t_1,\dots,t_k$ being given. Assume that $u_i^M\neq 1$ and $t_a^M\neq u_i^M$ for all $a=1,\dots,k\,.$ It is easy to check that under these assumptions system (3.5) can have solutions only if $\kappa^M=q_0^{2M(\ell_1+\ldots+\ell_N)}$, and one has $x_{ir}\neq x_{i,r+1}$ for all $r=0,\dots,M\,.$

Denote

$$
y_r\;=\;\frac{\mathcal{A}(u_i\,q_0^{2r-2})}{\kappa\,\mathcal{D}(u_i\,q_0^{2r-2})}\;\prod_{a=1}^{k}\frac{u_i\,q_0^{2r-2}-t_a\,q_0^{2}}{u_i\,q_0^{2r}-t_a}\;.
$$

Then $x_{i,r+1} - x_{ir} = y_1 y_2 \dots y_r (x_{i1} - x_{i0})$ and

$$(x_{i1} - x_{i0}) \sum_{r=0}^{M-1} y_1 y_2 \dots y_r = x_{iM} - x_{i0} = -2M \,,$$

which implies that the general solution of the system in question has the form

$$x_{ir} = x_{i1} - 2M \sum_{s=1}^{r-1} y_1 y_2 \dots y_s \Bigl(\sum_{s=0}^{M-1} y_1 y_2 \dots y_s \Bigr)^{-1} , \qquad r = 2, \dots, M \,,$$

with arbitrary x_{i1}. If $\sum_{r=0}^{M-1} y_1 y_2 \dots y_r = 0$, then system (3.5) is not solvable.

Thus, under the natural assumptions the solvability of system (3.5) implies that $\kappa = q_0^{2(p-\ell_1+\ldots+\ell_N)}$ for a certain integer p. In this case the solution of system (3.5) has the form

$$x_{ir} = c_i + x_r^{(p)}(u_i; t_1, \dots, t_k) \,, \qquad r = 1, \dots, M \,, \qquad i = 1, \dots, m \,,$$

cf. (3.2), with arbitrary $c_1, \dots, c_m$, while the vector $\| t_1, \dots, t_k; u_1, \dots, u_m; X_1, \dots, X_m \rangle\rangle$ does not depend on $c_1, \dots, c_m$ and equals $\| t_1, \dots, t_k; u_1, \dots, u_m \rangle\rangle_p$.

Remark. It has been shown in [DFM] that there is an action of the loop algebra $\widetilde{\mathfrak{sl}_2}$ in the space of states of the six-vertex model at roots of unity, and this action commutes with the transfer-matrix of the six-vertex model. The existence of a large symmetry algebra causes the degeneration of the spectrum of the transfer-matrix. As mentioned in [FM2] the symmetry degeneration of the spectrum of the transfer-matrix of the six-vertex model apparently corresponds to that degeneration of the spectrum which occurs in this case due to Proposition 3.1. Since Proposition 3.1 remains true and for the inhomogeneous arbitrary spin *XXZ* model with suitable quasiperiodic boundary conditions, one may suppose that the model has a large symmetry algebra in the general case too. Moreover, it could happen that an action of this algebra exists for any quasiperiodic boundary conditions, but commutes with the transfer-matrix only for special values of the quasiperiodicity parameter.

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